Math 121   <u>2.2 Rates of Change, Slopes, and Derivatives</u>

Objectives

1) Calculate average rate of change (no limit)
2) Calculate instantaneous rate of change (limit)
3) Calculate slope of a secant line (no limit)
4) Calculate slope of a tangent line (limit)
5) Recognize that
   - average rate of change is the same concept as slope of secant line
   - instantaneous rate of change is the same concept as slope of tangent line
6) Write the equation of a tangent line
7) Find the derivative of a function
   * using the definition of the derivative *

   Note: These instructions mean <u>limits</u>. Later we will use shortcuts to find derivatives without taking limits.
8) Recognize and use different notations which all mean the derivative:

$$f'(x) = y'(x) = \frac{d}{dx}f(x)$$

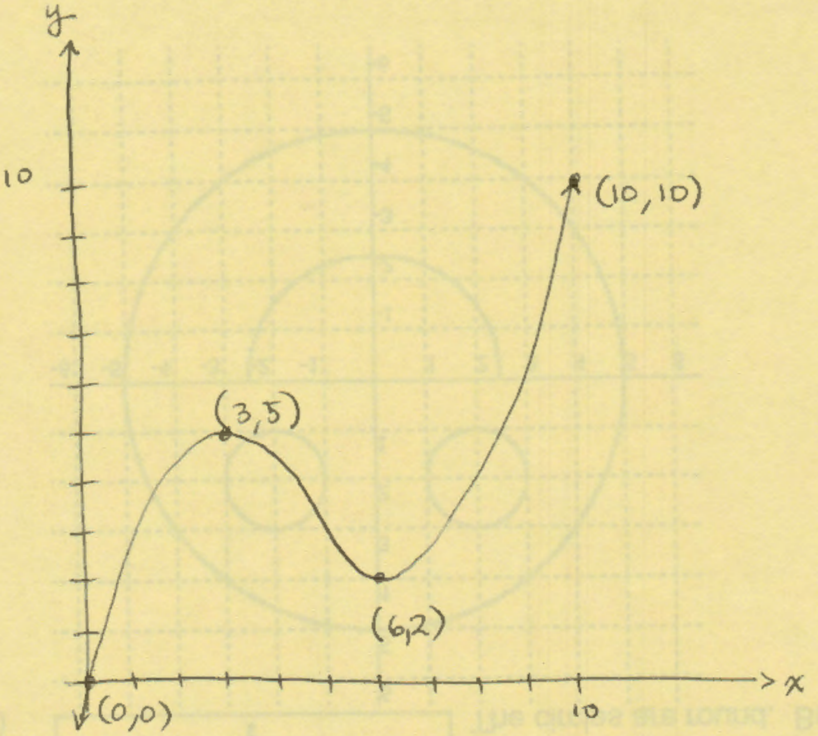
$$f' = y' = \frac{dy}{dx}$$

9) Use the words "differentiable" and "to differentiate" correctly. Also "non-differentiable".
   * in particular,

   "To find the derivative" = "to differentiate".

   "To derive" = "to use math to generate"

① a) Graph $f(x)=\frac{11}{126}x^3-\frac{155}{126}x^2+\frac{32}{7}x$ in GC with window $[0,10,1]\times[0,10,1]$.



Use TABLE to Notice that:

$f(0)=0$

$f(3)=5$

$f(6)=2$

$f(10)=10$

b) Find the average rate of change between $x=0$ and $x=3$.

average rate of change = slope of secant line $=\frac{y_2-y_1}{x_2-x_1}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$

$$=\frac{f(3)-f(0)}{3-0}$$

$$=\frac{5-0}{3-0}$$

$$=\boxed{\frac{5}{3}} \text{ or } \boxed{1.\overline{6}}$$

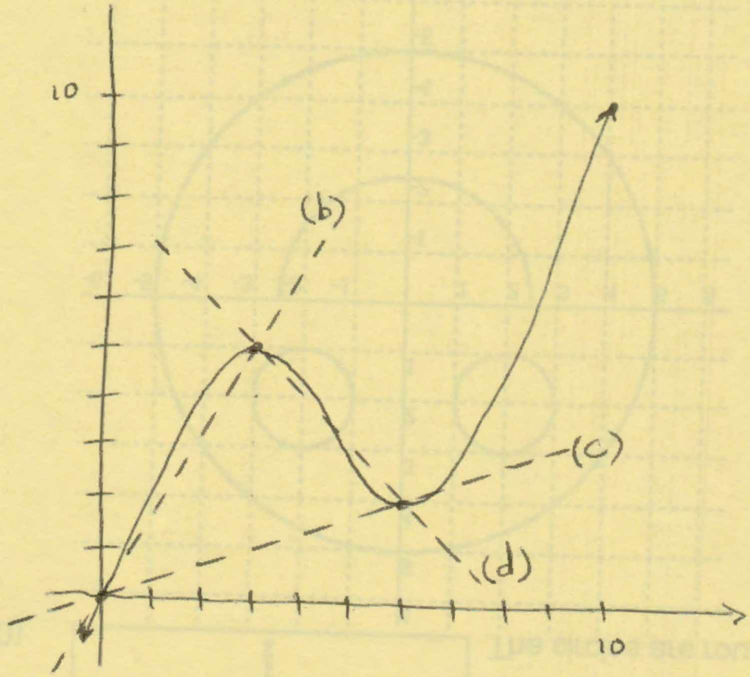
Note: This rate of change is positive because the function is generally increasing (going up) between $x=0$ and $x=3$.

c) Will the average rate of change between $x=0$ and $x=6$ be positive or negative?

positive — (0,0) is lower than (6,2). Graph goes up, overall. "on average"

d) Will the average rate of change between $x=3$ and $x=6$ be positive or negative?

$\boxed{\text{negative} \quad (3,5) \text{ is higher than } (6,2) \quad \text{Graph goes down on this interval}}$



e) Sketch the three secant lines from
- b) $x=0$ to $x=3$
- c) $x=0$ to $x=6$
- d) $x=3$ to $x=6$

These secant lines give an approximate idea of whether the graph goes up or down, but we'll want to know a more accurate rate of change, at a single point.

This is the slope of the tangent line, a line which touches the graph at only one point.

* We cannot use the slope formula with only one point *

WE MUST USE LIMITS.

A secant line cuts through two points on the graph.

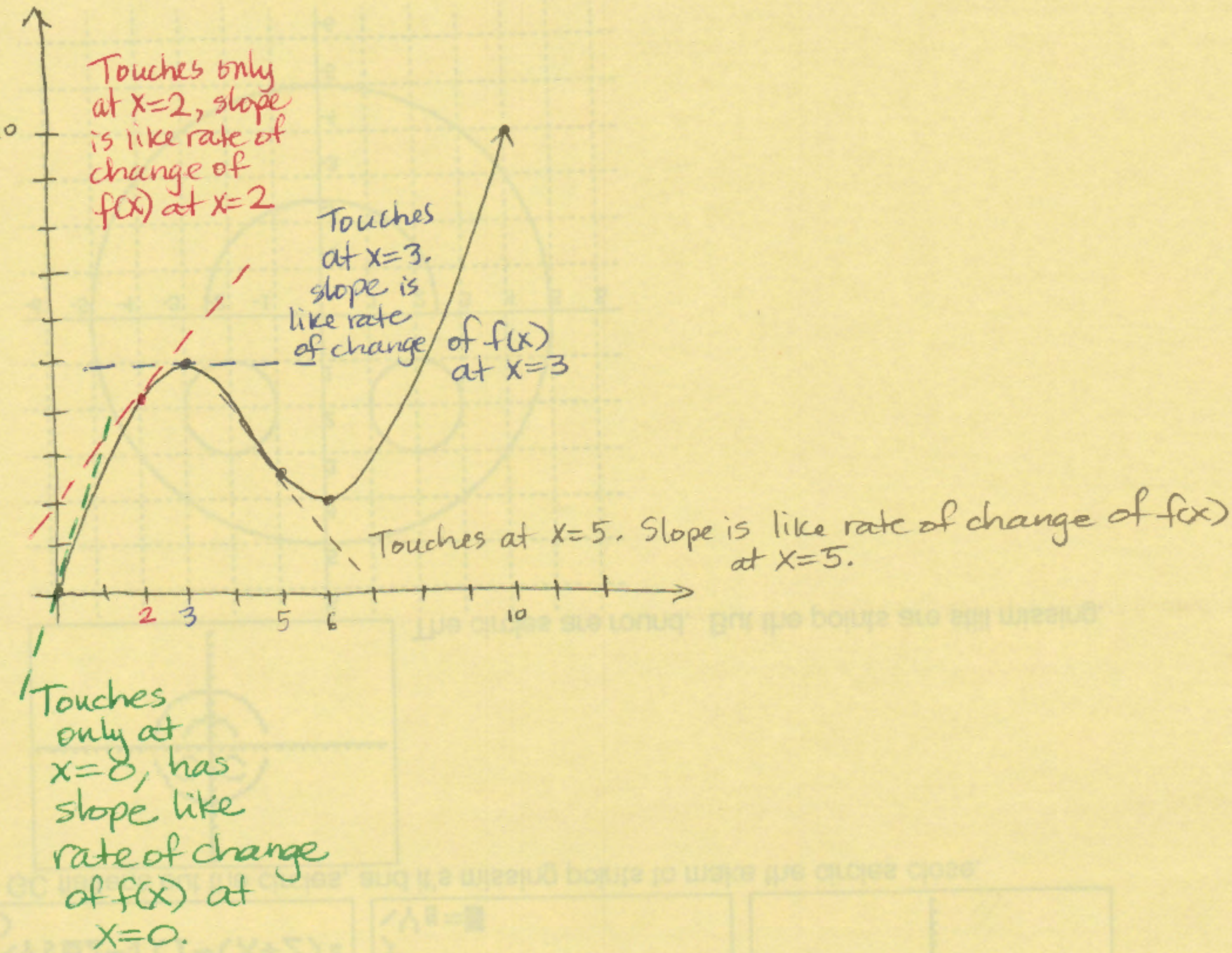
A tangent line touches at one point.

e) Sketch tangent lines at $x=0$

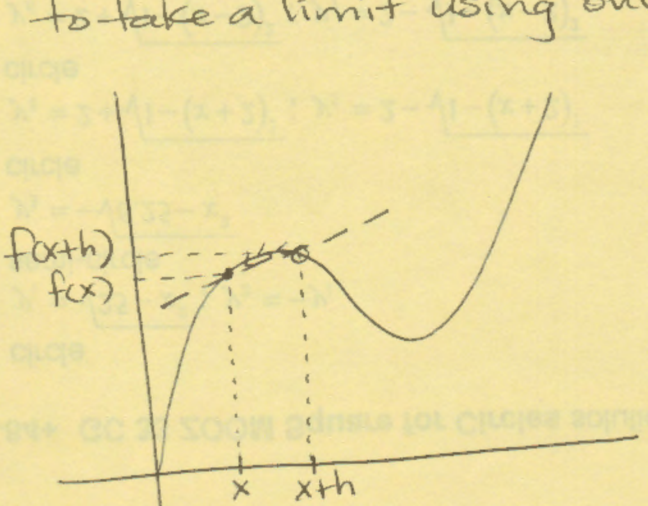
$x=2$

$x=3$

$x=5$



To find the slope of a tangent line, we need more flexible notation than $\frac{y_2 - y_1}{x_2 - x_1}$, because we need to take a limit using only one variable.



Let x be the x-coordinate of the point of tangency. So $f(x)$ is its y-coordinate.

Nearby is another x-coordinate that we get by adding h to $x \rightarrow (x+h)$.
Its y-coordinate is $f(x+h)$.

The slope of the line between these two points is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

h is the distance between the x-coordinates

$$= \frac{f(x+h) - f(x)}{x+h - x}$$

$$\boxed{m_{sec} = \frac{f(x+h) - f(x)}{h} = \text{slope of secant line} = \text{averager rate of change}}$$

The slope of the secant line approaches the slope of the tangent line as we take a limit which moves $(x+h) \longrightarrow x$, or numerically as $h \to 0$.

$$\boxed{m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{slope of tangent line} = \text{instantaneous rate of change at } x.}$$

* Remember we did the <u>difference quotient</u>?

$\frac{f(x+h) - f(x)}{h}$ is slope of secant $= m_{sec}$
$= \text{average rate of change}$

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is slope of tangent $= m_{tan}$
$= \text{instantanteous rate of change}$

Both use the difference quotient

f) Find the slope of the tangent line at $x=0$.

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

$$f(h) = \frac{11}{126}h^3 - \frac{155}{126}h^2 + \frac{32}{7}h$$

$$f(0) = \frac{11}{126}(0)^3 - \frac{155}{126}(0)^2 + \frac{32}{7}(0) = 0.$$

$$= \lim_{h \to 0} \frac{\frac{11}{126}h^3 - \frac{155}{126}h^2 + \frac{32}{7}h}{h}$$

$$= \lim_{h \to 0} \left(\frac{11}{126}\frac{h^3}{h} - \frac{155}{126}\frac{h^2}{h} + \frac{32}{7}\frac{h}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{11}{126}h^2 - \frac{155}{126}h + \frac{32}{7} \right)$$

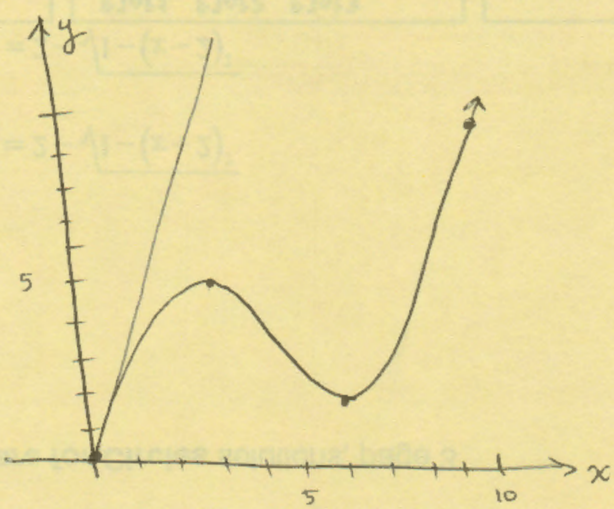
$$= \frac{11}{126}(0)^2 - \frac{155}{126}(0) + \frac{32}{7}$$

$$= \boxed{\frac{32}{7}}.$$

g) Find the equation of the tangent line to $f(x)$ at $x=0$ and add it to graph in GC.

Equation of a line: $m = \frac{32}{7}$ passing through $(0,0)$ y int!

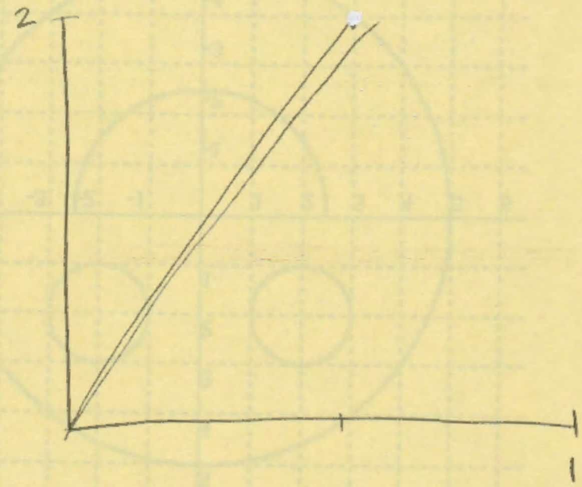
$$\boxed{y = \frac{32}{7}x}$$



h) ZOOM IN To see that $y = \frac{32}{7}x$ touches $f(x)$ only at $(0,0)$ and as we ZOOM in, the tangent line starts to look like $f(x)$ for x-values very close to $x=0$.

WINDOW

$[0,1,.1] \times [0,2,.1]$



We found the slope of the tangent line at $x=0$ by substituting $x=0$ in the first step.

But... in calculus, we want to know the slope of the tangent lines for all values of x, not just one.

If we leave x as a variable, we get a function, called the derivative (or the first derivative) into which we can plug any x-value and learn the slope of the tangent line to the function at that x-value.

The derivative has several notational names because calculus was simultaneously invented by several different people.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = y'(x) = \frac{d}{dx}f(x)}$$

Definition of the (first) Derivative

We say $f'(x)$ "f-prime of x"

$y'(x)$ "y-prime of x"

$\frac{d}{dx}f(x)$ "d-d-x of f of x"

So let's do this for an easier function.

(2) $f(x) = 3x^3 + x^2$

a) Find the derivative $f'(x)$ using the definition of the derivative.

b) Find the instantaneous rate of change of $f(x)$ when $x=2$.

c) Write the equation of the tangent line at $x=2$.

d) Graph $f(x)$ and tangent line from c) in GC.

Definition of derivative $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

evaluate $f(x+h)$ $= \lim_{h\to 0} \frac{3(x+h)^3 + (x+h)^2 - [3x^3 + x^2]}{h}$

do a lot of algebra $= \lim_{h\to 0} \frac{3(x^3+3x^2h+3xh^2+h^3) + (x^2+2xh+h^2) - 3x^3 - x^2}{h}$

$(x+h)^3$

$= (x+h)\underbrace{(x+h)(x+h)}$

$= (x+h)(x^2+2xh+h^2)$

$= x^3 + 2x^2h + xh^2$
$\quad + x^2h + 2xh^2 + h^3$

$= x^3 + 3x^2h + 3xh + h^3$

Distribute and combine $= \lim_{h\to 0} \frac{3x^3 + 9x^2h + 9xh^2 + 3h^3 + x^2 + 2xh + h^2 - 3x^3 - x^2}{h}$

0 0

*remember: With a difference quotient, anything w/o an h should add to 0.

$= \lim_{h\to 0} \left(\frac{9x^2h}{h} + \frac{9xh^2}{h} + \frac{3h^3}{h} + \frac{2xh}{h} + \frac{h^2}{h}\right)$

$= \lim_{h\to 0} (9x^2 + 9xh + 3h^2 + 2x + h)$

$= 9x^2 + 9x(0) + 3(0)^2 + 2x + 0$

$= 9x^2 + 2x$

$\boxed{f'(x) = 9x^2 + 2x}$

b) instantaneous rate of change at $x=2$

$$f'(2) = 9(2)^2 + 2(2)$$
$$= 36+4$$
$$= \boxed{40}$$

c) equation of tangent line at $x=2$.

Touches $f(x)$ at $x=2$, so y-coordinate on tangent line is the same y-coordinate as y-coordinate on function.

$$f(2) = 3(2)^3 + 2^2$$
$$= 24+4$$
$$= 28$$

Write equation of line through $(2,28)$ with slope $= 40$ (from part b).

$$y-28 = 40(x-2)$$
$$y = 40x - 80 + 28$$
$$y = 40x - 52$$

d) $\boxed{Y=}$

$Y_1 = 3x^3 + x^2$

$Y_2 = 40x - 52$

WINDOW $[0, 3, 1] \times [0, 35, 5]$